Trigonometry

• Trigonometry is the study of triangles – the relationship between their sides and angles.

• Oddly enough our study of triangles begins with a circle.

  The unit circle is a circle with a radius equal to one.

  Notice that if we rotate upward from the x-axis an amount $\theta$ to a point $P(x,y)$ on the circle and draw a line (terminal ray), we can construct a triangle.

  The “legs” of the triangle change with rotation angle.
We define the radius (terminal ray) of the circle to be the **hypotenuse** of the triangle formed.

As this hypotenuse rotates through the circle, the length of the sides opposite and adjacent to this angle change in a prescribed way.

What the Greeks did, was systematically measure this variation and create three ratios to describe it.

These ratios they termed **sine**, **cosine**, and **tangent**

\[
\begin{align*}
\sin(\theta) &= \frac{\text{opposite}}{\text{radius}} = \frac{\text{opposite}}{\text{hypotenuse}} \\
\cos(\theta) &= \frac{\text{adjacent}}{\text{radius}} = \frac{\text{adjacent}}{\text{hypotenuse}} \\
\tan(\theta) &= \frac{\text{opposite}}{\text{adjacent}}
\end{align*}
\]
Trigonometry

• Why ratios? Why not just define the relationship between the angle and the various sides of the triangle?

   Like \(\sin(\theta) = \text{vertical length} \), \(\cos(\theta) = \text{horizontal length} \), and \(\tan(\theta) = \text{hypotenuse length} \)

• Using ratios of the various sides allows the relationship to extend to any size circle (any length hypotenuse or leg).

Trig functions defined in terms of ratios give the same result for a given angle regardless of the size of the radius

\[ \theta = 45^\circ \rightarrow \sin(45^\circ) = \frac{0.707}{1} \]

\[ \theta = 45^\circ \rightarrow \sin(45^\circ) = \frac{1.414}{2} \]
Great, but how does knowing about a relationship developed by some ancient guys help me with physics?

First, we need to understand that many physical quantities are what we call **vectors**.

A vector is any measurable quantity where direction is important.

- Consider the following: Your friend calls and asks you to meet her two miles from the Westmoreland mall in half an hour.
- There are two measurable (physical) quantities that are important in this scenario, the time (when) and the location (where) you will meet.
- Since time has only one direction it need not be specified. Time is a **scalar** quantity.
- Position on the other hand is a **vector** and requires direction to get it right.
Trigonometry and Physics

Position or location requires that you not only specify a magnitude (2 mi.), but a direction from a reference point.

There are a lot of places two miles from the mall, but only one that is two miles, 15 degrees southwest of the mall.
Notice that the displacement (change in location) is the radius (red) of a circle (2 mi).

If I extend a vertical line (green) from the east-west axis down until it meets the radius, this gives me my southward motion and the vertical leg of a triangle.

If I now extend a line (blue) from the origin, along the east-west axis, to the point on the east-west axis that the vertical line originated, I get the westward motion and the final leg of the triangle.
Trigonometry and Physics

- We know three things about the triangle on the previous page and reproduced below
  - We know the length of the hypotenuse (displacement vector) – 2 mi.
  - We know the angle made between the displacement vector and the horizontal vector – 15°
  - We know the angle between the horizontal and vertical vectors – 90°
- This is what is referred to as an angle-angle-side (AAS) configuration, and from it we can determine the lengths of the other two sides and the remaining angle.
Starting with the unknown angle, you should all remember from geometry that the sum of the interior angles of any triangle is 180°. So if two of those angle, 15° and 90° add to 105°, the final angle must be 180° – 105° or 75°.

Next, we can apply the trigonometric relationships (ratios) previously developed to solve for the remaining sides.

While we solved for angle θ first, we could have started with any unknown quantity, the point is, if we know any three quantities we can solve for the remaining three.
Useful Trigonometric Relationships

- **Law of Sines.**
  \[
  \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
  \]

- **Law of Cosines.**
  \[
  c^2 = a^2 + b^2 - 2ab \cos(C)
  \]

- Notice that if angle \( C \) is 90° (\( \cos(90°) = 0 \)) the law of cosines reduces to the more familiar form of the Pythagorean equation.

  **Pythagorean equation ONLY** works with right triangles whereas the **Law of Cosines** works with any triangle.
Let’s say you took a hike and set-off heading north, walking four kilometers. At that point you stopped for a drink and short rest. When you decided to continue, you veered right and walked three kilometers due east where you stopped for lunch. At the location you took lunch, how far had you walked? How far were you from your starting point?

The total distance walked is the sum of the distance walked during each interval, which in this case is 4 km and 3 km for a total of 7 km (4 km + 3 km).

The distance from the start—called the displacement—requires trigonometry.
Let’s Look at an Example

- To find the displacement, we first recognize that the pattern of our motion forms the two “legs” of a right triangle with our displacement (green) being the hypotenuse.
- From the Law of Cosines we can write: $c^2 = a^2 + b^2 - 2ab \cos(C)$, but $C = 90^\circ$ so that $c$, the length of our displacement reduces to: $c^2 = a^2 + b^2$

$\begin{align*}
c &= \sqrt{a^2 + b^2} = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5\text{ km}
\end{align*}$

- From the Law of Sines we determine our baring, $B$: $\frac{\sin(C)}{c} = \frac{\sin(B)}{b}$
Let’s Look at an Example

- From the Law of Sines, we rearrange the equation solving for \( \sin(B) \).

\[
\sin(B) = \frac{b \sin(C)}{c}
\]

- We need to undo the sine function, for that we use the arcsine or \( \sin^{-1} \).

\[
\sin^{-1}(\sin(B)) = \sin^{-1}\left(\frac{b \sin(C)}{c}\right)
\]

- **NOTE** \( \sin^{-1}(x) \) is **not** the same as \( \frac{1}{\sin(x)} \)

- The arcsine undoes the sine so that \( \sin^{-1}(\sin(x)) \) is just \( x \)

\[
B = \sin^{-1}\left(\frac{b \sin(C)}{c}\right) = \sin^{-1}\left(\frac{3 \sin(90\degree)}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right) = 36.87\degree
\]
Let’s Look at an Example

As a check, let’s use the definition of the cosine of an angle to evaluate our result for $B$.

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

or,

$$\text{adjacent} = \text{hypotenuse} \cdot \cos(\theta)$$

$$\text{adjacent} = 5 \text{ km} \cdot \cos(36.87^\circ) = 4 \text{ km}$$

This is indeed consistent with the value of 4 km stated in the problem.
Let’s Look at another Example

• Let’s say you took a hike and set-off heading north, walking four kilometers. At that point you stopped for a drink and short rest. When you decided to continue, you veered right and walked three kilometers 15° north of east where you stopped for lunch. At the location you took lunch, how far had you walked? How far where you from your starting point?

• As before, you’ve walked a total distance of 7 km

• Your displacement however is not 5 km. It’s more.
Looking at this other Example

- Observe there are two right triangles one with a hypotenuse given by $b$ one with a hypotenuse given by $c$
Looking at this other Example

- Notice that the base of the green triangle, \( b' \), is the same length as the base of the red one.

- Notice that the height of the green triangle \( a' \) is the sum of the length of vector \( a \) and the height of the red triangle.

\[
c^2 = a'^2 + b'^2 - 2(a'b')\cos(90°)
\]

\[
a' = a + b \cdot \sin(15°) \quad b' = b \cdot \cos(15°)
\]

\[
c^2 = (a + b \cdot \sin(15°))^2 + (b \cdot \cos(15°))^2 - 2(a + b \cdot \sin(15°))(b \cdot \cos(15°))\cos(90°)
\]
Looking at this other Example

\[ c = \sqrt{(a + b \cdot \sin(15^\circ))^2 + (b \cdot \cos(15^\circ))^2 - 2(a + b \cdot \sin(15^\circ))(b \cdot \cos(15^\circ)) \cos(90^\circ)} \]

\[ c = \sqrt{(4 + 3 \cdot \sin(15^\circ))^2 + (3 \cdot \cos(15^\circ))^2 - 2(4 + 3 \cdot \sin(15^\circ))(3 \cdot \cos(15^\circ)) \cos(90^\circ)} \]

\[ 0 \text{ since } \cos(90^\circ) = 0 \]

\[ c = \sqrt{(4 + 3 \cdot \sin(15^\circ))^2 + (3 \cdot \cos(15^\circ))^2} = \sqrt{(4 + 0.78)^2 + (2.90)^2} = 5.59 \text{ km} \]

As expected, your displacement is greater than 5 km
Graphical Look at Trig Functions

\( x \) — in degrees

NOTICE \( \sin(90^\circ) = 1 \)

NOTICE \( \cos(90^\circ) = 0 \)